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Semilinear Evolution Inclusions in Banach spaces

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Abstract

We study nonautonomous semilinear evolution inclusions in Banach spaces. Existence of limit solutions is proved. Variant of Filippov–Pliss lemma and relaxation theorem are also obtained.

Keywords: *nonautonomous semilinear evolution, Banach spaces*

1 Introduction

In this paper we study a class of parabolic semilinear differential inclusions, having the form

$$\begin{cases} \dot{y} \in A(t)y + F(t,x), t \in I = [t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (1.1)$$

We refer the reader to the books [7, 9] for the general theory of such kind of systems.

Evolution inclusion(1.1) is studied in the literature under some compactness or dissipative type assumptions.

Assume that the evolution operator $K(\cdot,\cdot)$ is compact. In this case one has to use some conditions of the image of $F(\cdot,\cdot)$ has weak compactness of the values or the Banach space Y is assumed to be reflexive. We refer the reader to [2, 11]. f such conditions are not assumed then it is impossible practically to prove existence of mild solution. Notice also [10, 18], where measure of noncompactness is used.

In this paper we assume that $A(\cdot)$ generates a compact operator $K(\cdot,\cdot)$, however, the values of $F(\cdot,\cdot)$ are only closed bounded. We define a new type of a solutions (limit solutions) which have similar properties of the mild solutions. The advantage here is that the limit solution set is nonempty and compact.

The dissipative type of assumptions are used commonly when $A(\cdot)$ generates an equicontinuous operator and $F(\cdot,\cdot)$ is (locally) Lipschitz or more general locally Perron. In this case the solution set is nonempty but not closed even $F(\cdot,\cdot)$ has closed convex and bounded values. Its closure is the set of the limit solutions defined and studied here.

In the last section of this paper we use one sided Perron (OSP) assumptions and prove also the existence of limit solution in case $F(\cdot,\cdot)$ almost continuous but $A(\cdot)$ generate only equicontinuous operator $K(\cdot,\cdot)$. If the operator is also compact that evidently there exist integral solutions when $F(\cdot,\cdot)$ admits convex values. We do not assume that F has convex values and prove that the set of mild solutions is dense in the set of limit solutions (a form of Relaxation theorem).

Let Y ba Banach space with dual Y^* and let $A \subset Y$ be nonempty, closed and bounded. The function

$$\sigma(l, A) = \sup_{a \in A} \langle l, a \rangle,$$

where $l \in Y^*$ is called support function (of A).

Let $A, B \subset Y$ the Hausdorff distance between A and B is defined by

$$D_H(A, B) = max\{Ex(A, B), Ex(B, A)\}$$

where $Ex(A, B) = \sup_{b \in B} \inf_{a \in A} d(a, b)$.

Definition 1.1 *The multifunction $F: I \times Y \rightrightarrows Y$ is said to be lower semicontinuous (LSC) if for any $(t, y) \in I \times Y$, any $v \in F(t, y)$ and any sequence $(t_n, y_n)_n$ with $t_n \to t$ and $y_n \to y$ there exists a sequence $(v_n)_n$ with $(v_n) \in F(t_n, y_n)$ such that $v_n \to v$. Further $F(\cdot,\cdot)$ is continuous if it is continuous w.r.t the Hausdorff metric.*

$F(\cdot,\cdot)$ is called upper hemicontinuous if the support function $\sigma(l, F(\cdot,\cdot))$ is upper semicontinuous as real valued function.

The multifunction $F: I \times Y \rightrightarrows Y$ is said to be almost lower semicontinuous (almost continuous, etc) if for any $\varepsilon > 0$ there exists a compact set $I_\varepsilon \subset I$ with meas$(I \backslash I_\varepsilon) < \varepsilon$ such that $F(\cdot,\cdot)$ is lower smicontinuous (continuous, etc) on $I_\varepsilon \times Y$.

Definition 1.2 *Let E be a closed subset of Y, w be a real number and $\Delta = \{ (t,s): 0 \leq s \leq t\}$. A family of mappings*

$$K = \{K(t,s): E \to E \ ; 0 \leq t \leq s\}$$

is said to be an evolution operator of type w if the following properties are satisfied:

(i)$K(t,t) = I$ for all $x \in E$ and $t \geq 0$;

(ii)$K(t,s)K(s,r) = K(t,r)$ for all $x \in E$ and $0 \leq r \leq s \leq t$;

(iii) For each $x \in E$, the mapping $(t,s) \to K(t,s)x$ is continuous from Δ into Y;

(iv) $\| K(t,s)x - K(t,s)y \| \leq \| x - y \| e^{w(t-s)}$ for all $x, y \in E$ and $0 \leq s \leq t$.

For every evolution system, we can consider the respective evolution operator $K: \Delta \to \mathcal{L}(Y)$, where $\mathcal{L}(Y)$ is the space of all bounded linear operators in Y. Since the evolution operator K is strongly continuous on the compact set Δ, by the uniform boundedness theorem there exist a constant $D = D_\Delta > 0$ such that

$$\| K(t,s) \|_{\mathcal{L}(Y)} \leq D, \quad (t,s) \in \Delta.$$

Recall that the evolution operator is said to be compact when $K(t,s)$ is a compact operator for all $t - s > 0$ i.e. $K(t,s)$ maps bounded sets into relatively compact sets.

Definition 1.3 *Let $x, y \in Y$, where Y is a real Banach space with norm $\|\cdot\|$. The right directional derivative of the norm of x in the direction y is defined as*

$$[x,y]_+ = \lim_{h \to 0^+} \frac{\|x+hy\| - \|x\|}{h}.$$

It is well known that

(i) $|[x,y]_+ - [x,z]_+| \leq \| y - z \|$

(ii) $[\cdot,\cdot]_+$ is upper semicontinuous as a real valued function.

2 Limit Solutions

In this section we define the limit solutions of (1.1) introduced in [4] and study their main properties.

Let Y be a Banach space and $I = [t_0, T] \subset \mathbb{R}$. We consider (1.1), i.e.

$$\begin{cases} \dot{y} \in A(t)y + F(t, y(t)), & t \in I \\ y(t_0) = y_0. \end{cases}$$

Here $\{A(t)\}_{t\in[t_0,T]}$ is a family of densely defined linear operators which generates a strongly continuous evolution operator $K: \Delta \to \mathcal{L}(Y)$, and $F: I \times Y\tilde{A}Y$ is a multivalued map with nonempty closed values.

Definition 2.1 *The continuous function $z(\cdot)$ is said to be a (mild) solution of (1.1) if*

$$z(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)f_z(s)ds,$$

where $f_z(\cdot)$ is a measurable selection of $F(\cdot, z(\cdot))$, i.e. $f_z(t) \in F(t, z(t))$.

We will call $f_z(\cdot)$ a pseudo derivative of $z(\cdot)$.

We will use the following assumptions:

A1. $\{A(t)\}_{t\in[t_0,T]}$ is the family of densely defined linear operators which generates a strongly continuous evolution operator $K: \Delta \to \mathcal{L}(Y)$.

A2. The operator $K(t, s)$ is compact for all $t > s$.

F1. $F(\cdot,\cdot)$ satisfies a growth condition, i.e. there exist a constant (Lebesgue integrable) C such that$|F(t, y)| \leq C(1 + |y|)$.

F2. $F(\cdot, y)$ is measurable.

Definition 2.2 *The continuous function $y: I \to Y$ is called ε-solution of* (1.1) *if it is a solution on I of the problem*

$$\begin{cases} \dot{y} \in A(t)y + F(t, y + \varepsilon\mathbb{B}), \\ \quad y(t_0) = y_0, \quad t \in I. \end{cases} \tag{2.1}$$

Lemma 2.3 *Under **A1, F1, F2** there exist two constants M and N such that $|y(t)| \leq M$ and $|F(t, y(t) + \mathbb{B})| \leq N - 1$ for every solution $y(\cdot)$ of*

$$\dot{y}(t) \in A(t)y + \overline{co} \quad F(t, y(t) + \mathbb{B}) + \mathbb{B}. \tag{2.2}$$

Lemma 2.4 *Under **A1, F1, F2** for every $\varepsilon > 0$ there exist ε-solution of (1.1). If **A2** also hold then the set of all ε-solutions is $C(I, Y)$ precompact.*

Proof. Let $f_0(t) \in F(t, y_0)$ be measurable selection. We define

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)f_0(s)ds, \quad t \in [t_0, T].$$

We have that

$$|y(t) - y_0| \leq |K(t, t_0)y_0 - y_0| + Me^{w(t-t_0)}C(1 + |y_0|)(t - t_0) \tag{2.3}$$

for any $t \in [t_0, T]$. Hence, for every $\varepsilon > 0$ there exist $t_1 > t_0$ such that $|y(t) - y_0| < \varepsilon$ for any $t \in [t_0, t_1]$, i.e.

$$F(t, y_0) \subset F(t, y(t) + \varepsilon\mathbb{B}).$$

Suppose that the required ε-solution $y(\cdot)$ exists on $[t_0, \tau)$ with $\tau < T$. Due to Lemma 2.3 $\lim_{tZ\tau} y(t) = y_\tau$ exist. We study (2.1) on $[\tau, T]$ replacing t_0 by τ and y_0 by y_τ. Dealing as before we can show that there exist $\tau_1 > \tau_0$ such that

$$y(t) = K(t, \tau)y_\tau + \int_{\tau}^{t} K(t, s)f_y(s)ds, \quad f_y(t) \in F(t, y(t))$$

is ε-solution on $[\tau, \tau_1]$. Using trivial modification of Zorn's Lemma one can show that $y(\cdot)$ exists on $[t_0, T]$. Let $y(\tau) = y_\tau$. Replacing y_0 by y_τ and t_0 by τ in (2.3) we derive

$$|y(t) - y_\tau| \leq |K(t, \tau)y_\tau - y_\tau| + Me^{w(t-\tau)}C(1 + |y_\tau|)(t - \tau)).$$

Therefore there exists ε-solution. Let **A2** hold. Since $K(t,s)$ is strongly continuous and compact, one has that it is equicontinuous. If $\{y^n(\cdot)\}_{n=1}^{\infty}$ is a sequence of ε-solutions, then it is equicontinuous. Furthermore $K(t,s)(N\mathbb{B})$ is precompact and hence Arzela-Ascoli theorem applies. Consequently $\{y_n(\cdot)\}_{n=1}^{\infty}$ is precompact and hence the set of ε-solutions is precompact, i.e. its closure is $C(I,Y)$ compact set.

Remark 2.5 *Notice that the conclusion of Lemma 2.4 holds also for $\varepsilon = 1$. Furthermore $F(t,x) + \mathbb{B}$ also satisfies the conditions of Lemma 2.3. Therefore the solution set of (2.2) is precompact.*

Definition 2.6 *The continuous function $y: I \to Y$ is called a limit solution of (1.1) if there exist a sequence of positive numbers $(\varepsilon_n)_n \to 0$ and ε_n-solution $y^n(\cdot)$ such that $lim_{n\to\infty} y^n(t) = y(t)$ uniformly on I.*

Theorem 2.7 *The limit solution set of (1.1) is nonempty and compact.*

Proof. Let $\{y_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of ε_n-solutions with $\varepsilon_n \downarrow 0$. Clearly every ε_n-solutions is also ε_1-solution ($\varepsilon_1 > \varepsilon_n$). From Lemma 2.4 we know that the set of ε_1-solutions is precompact and hence there exist a uniformly converging subsequence $\{y_{n_k}\}_{n=1}^{\infty}$ with limit $y(\cdot)$. This function $y(\cdot)$ is a limit solution. The set of limit solutions is a subset of the closure of ε_1-solutions and hence it is a closed subset of a compact set i.e. it is itself compact.

Recall that $f(t,\cdot)$ is said to be locally Lipschitz if for every $y \in Y$ there exist a neighborhood U of y and $U \subset Y$ such that f restricted to $I \times U$ is Lipschitz.

Proposition 2.8 *Let $f(\cdot,\cdot)$ be Caratheodary with $f(t,\cdot)$ locally Lipschitz. If $f(\cdot,\cdot)$ satisfies H_1 and A_1, then the evolution equation*

$$\begin{cases} \dot{y} = A(t)y + f(t, y(t)), \\ \quad y(t_0) = \eta, \quad t \in I. \end{cases} \tag{2.4}$$

has a unique solution $y(\cdot,\eta)$, which is defined on I and depends continuously on the initial condition.

Along with (1.1) we consider the system

$$\begin{cases} \dot{y} \in A(t)y + W(t,y), \\ \quad y(t_0) = y_0, \end{cases} \tag{2.5}$$

Where $W(t,y) = \bigcap_{\varepsilon>0} \overline{co} \quad F(t, y + \varepsilon\mathbb{B})$.

Definition 2.9 *Let Y be a complete metric space.*

(i) A set $A \subset Y$ is said to be contractible if there exists a continuous function $H: [0,1] \times A \to A$ and $\tilde{y} \in A$ such that $H(0,y) = y$ and $H(1,y) = \tilde{y}$ on A.

(ii) The set $A \subset Y$ is said to be compact R_δ if there exist a decreasing sequence of compact contractible sets A_n such that $A = \bigcap_{n=1}^{\infty} A_n$.

Theorem 2.10 *Under the assumptions **A1, F1, F2** the limit solution set of (2.5) is nonempty R_δ.*

Proof. The proof will be given by locally Lipschitz approximations of W. Let us denote $\mathbb{LS}$ the set of limit solutions of (2.5). Let $r_n = \frac{1}{3^n}$ and $(V_\nu)_{\nu\in\mathcal{M}}$ be a locally finite refinement of the open covering $Y = \bigcup_{y\in Y} (y + r_n\mathbb{B})$. Let $(\psi_\nu)_{\nu\in\mathcal{M}}$ be a locally Lipschitz partition of unity subordinate to $(V_\nu)_{\nu\in\mathcal{M}}$ and take y_ν such that $V_\nu \subset y_\nu + r_n\mathbb{B}$. Consider the approximations

$$W_n(t,y) = \sum_{\nu\in\mathcal{M}} \psi_\nu(y) C_\nu(t),$$

where $C_\nu(t) = W(t, y_\nu + 2r_n\mathbb{B})$. Then we have that

$$W(t,y) \subset W_{n+1}(t,y) \subset W_n(t,y) \subset W(t, y + 3r_n\mathbb{B}) \quad on \;\; I \times Y \tag{2.6}$$

Denote by S_n the mild solution of

$$\begin{cases} \dot{y} \in A(t)y + W_n(t,y), \\ \quad y(t_0) = y_0, \end{cases} \tag{2.7}$$

By (2.6) we have $S_{n+1} \subset S_n$. Moreover, the solution set S_n is compact for every n with $r_n < \frac{1}{3}$. We shall prove that S_n is contractible. Let $\tilde{f}_\nu$ be a measurable selection of $F(\cdot, y_\nu)$ for every $\nu \in \mathcal{M}$. We define

$$f(t,y) = \sum_{\nu \in \mathcal{M}} \psi_\nu(y) \tilde{f}_\nu(t) \;\; on \;\; I \times Y.$$

Since for a.e. $t \in I$, $\tilde{f}_\nu \in F(t, y_\nu) \subset C_\nu(t)$, we have that $f(t,y) \in W_n(t,y)$ for a.e. $t \in I$ and $y \in Y$. Since $(V_\nu)\nu \in \mathcal{M}$ is locally finite we have $f(\cdot, y)$ is measurable and $f(t,\cdot)$ is locally Lipschitz. Due to Proposition 2.8 the equation

$$\begin{cases} \dot{y} \in A(t)y + f(t, y(t)), \; t \in [s,T] \\ y(s) = \rho, \end{cases} \tag{2.8}$$

has a unique solution $\tilde{y}(\cdot, s, \rho)$ which depend continuously on ρ. Take $\tau \in [0,1]$ and denote

$b_\tau = \tau(T - t_0) + t_0$. We define the homotopy $H: [0,1] \times \overline{S_n} \to \overline{S_n}$ as

$$H(\tau, v)(t) = \begin{cases} v(t), & t \in [t_0, b_\tau] \\ \tilde{y}(t; b_\tau, v(b_\tau)), & t \in (b_\tau, T] \end{cases}$$

Let $u(\cdot), v(\cdot) \in \overline{S_n}$ and let

$$\dot{z} = A(t)z + f(t, z(t)), \quad z(s) = u(s)$$

$$\dot{y} = A(t)y + f(t, y(t)), \quad y(\tau) = u(\tau).$$

Suppose $\tau > s$. Then

$$z(\tau) = K(\tau, s)u(s) + \int_\tau^s K(\tau, \mu) f(\mu, z(\mu)) d\mu.$$

Thus

$$z(t) = K(t, \tau) z(\tau) + \int_\tau^t K(t, \mu) f(\mu, z(\mu)) d\mu.$$

Consequently

$$y(t) - z(t) = K(t,\tau)(y(\tau) - z(\tau)) + \int_\tau^t K(t,\mu)[f(\mu, y(\mu)) - f(\mu, z(\mu))] d\mu,$$

i.e.

$$|y(t) - z(t)| \le |K(t,\tau)(y(\tau) - z(\tau))| + \int_\tau^t K(t,\mu) l(\mu) |y(\mu) - z(\mu)| d\mu.$$

Assume that $|u(t) - v(t)| \leq \varepsilon$. Fix $\delta > 0$ and $|\tau - s|$ so small that $|u(\tau) - u(s)| < \delta$ and $|K(t,s)u(\xi) - K(\tau,\tau)| < \delta$ for any $|\xi| \leq M$ and $ND(t-s) < \delta$.

Consequently

$$|z(\tau) - v(\tau)| \leq |v(\tau) - v(s)| + |z(\tau) - z(s)|$$

$$\leq \delta + |K(\tau,s)u(s)_K(\tau,\tau)u(s)| + |\int_s^\tau K(\tau,\mu)f(\mu,y(\mu))d\mu$$

$$\leq \delta + \delta + D.M(t-s) \leq 3\delta.$$

Let $|u(s) - v(s)| \leq \varepsilon \quad \forall s \in [t_0, T]$, then

$$|y(\tau) - z(\tau)| \leq |u(s) - v(s)| + |y(\tau) - u(s)| + |z(\tau) - z(s)|$$

$$\leq \varepsilon + 4\delta.$$

Consequently

$$|y(t) - z(t)| \leq |K(t,\tau)||y(\tau) - z(\tau)| + \int_\tau^t |K(t,s)|l(s)|y(s) - z(s)|ds,$$

i.e.

$$|y(t) - z(t)| \leq D(\varepsilon + 4\delta) + D\int_\tau^t l(s)|y(s) - z(s)|ds.$$

Thus $|u(t) - z(t)| \leq r(t)$, where $r(t) = D(\varepsilon + 4\delta) + D\int_\tau^t l(s)r(s)ds$. Gronwall's inequality then applies and hence $H(\cdot,\cdot)$ is continuous map from $[0,1] \times \overline{S_n} \to \overline{S_n}$. Furthermore $H(0,v) = \tilde{y}$ and $H(1,v) = v$. So, we find a decreasing sequence of compact contractible sets $(\overline{S_n})_n \subset C(I,Y)$. By Definition 2.9, we have to only show that

$$\mathbb{LS} = \bigcap_{n=1}^\infty \overline{S_n}.$$

Notice that $\mathbb{LS} \subset \overline{S_n}$ for any $n \in \mathbb{N}$. Let $y \in \mathbb{LS}$ and fix n. Then, there exist a decreasing $(\varepsilon_m)_m \downarrow 0$ and $(y_m)_m$ a sequence of ε_m-solution for (2.5) such that $y_m \to y$. Let m_n be such that $\varepsilon_{m_n} < r_n$. For any $m \geq m_n$ we have

$$W(t, y_m(t) + \varepsilon_m\mathbb{B}) \subset W(t, y_m(t) + r_n\mathbb{B}) \subset W_n(t, y_m(t)),$$

because, if $\psi_\nu(y_m(t)) > 0$, then $y_m(t) \in V_\nu \subset y_\nu + r_n\mathbb{B}$. Hence, $y_m \in S_n$, for any $m \geq m_n$. Now, let $s \in \bigcap_{n=1}^{\infty} \overline{S_n}$ so $s \in \overline{S_n}$, for any n. Then for any n there exist a sequence $(z_n^m)_m \subset S_n$ such that $z_m^n \to s(t)$ uniformly when $m \to \infty$. By (2.6) z_m^n is a solution of

$$\dot{y} \in A(t)y + W(t, y + 3r_n\mathbb{B}). \dot{y} \in A(t)y + W(t, y + 3r_n\mathbb{B}).$$

Thus z_m^n is $\varepsilon_n = 3r_n$-solution for $\dot{y} \in A(t)y + W(t,y)$. Let $s_n = z_n^n$. Then s_n is ε_n-solution of $\dot{y} \in A(t) + W(t,y)$ and $s_n(t) \to s(t)$ uniformly on I as $n \to \infty$, i.e. , s is a limit solution for $\dot{y} \in A(t)y + W(t,y)$.

Proposition 2.11 *Let **A1, F1, F2** hold true. If $F(\cdot,\cdot)$ is (jointly) $\mathcal{L} \otimes \mathcal{B}$ (Lebesgue, Borel) measurable then for any $\varepsilon > 0$ and every $\delta > 0$ the solution set of*

$$\begin{cases} \dot{y} \in A(t)y + \overline{co}\ F(t, y + \varepsilon\mathbb{B}), \\ \qquad y(t_0) = y_0 \end{cases} \tag{2.9}$$

is contained in the closure of the $(\varepsilon + \delta)$-solution set of (1.1).

Proof. Let $y(\cdot)$ be a solution of (2.9) with pseudo derivative $f_y(\cdot)$. That is

$$y(t) = K(t,t_0)y_0 + \int_{t_0}^{t} K(t,s)f_y(s)ds.$$

Furthermore, for any $t_0 \leq \tau \leq t \leq T$ one has that

$$y(t) = K(t,\tau)y(\tau) + \int_{\tau}^{t} K(t,s)f_y(s)ds.$$

Since $F(\cdot,\cdot)$ is measurable and $y(\cdot)$ is continuous one has that $t \to F(t, y(t) + \varepsilon\mathbb{B})$ is measurable.

$$\overline{\int_{\tau}^{t} K(t,s)\overline{co}\ F(s, y(s) + \varepsilon\mathbb{B})ds} = \overline{\int_{\tau}^{t} K(t,s)F(s, y(s) + \varepsilon\mathbb{B})ds}.$$

Fix $\nu > 0$. The solution set of (2.9) is equicontinuous, i.e. there exists a uniform subdivision

$$t_0 < t_1 < \ldots . < t_k < \ldots .. < t_n < t_{n+1} = T$$

such that $|y(t) - y(\tau)| < \frac{\nu}{5} \quad \forall t, s \in [t_k, t_{k+1}], \quad k = 0,1,2,3.\dots.n$. Clearly for any $[t_k, t_{k+1}]$ there exist a measurable selection $f_z(t) \in F(t, y(t) + \varepsilon\mathbb{B})$ such that

$$\left|\int_{t_k}^{t_{k+1}} K(t_{k+1}, s)|f_y(s) - f_x(s)|ds\right| < \frac{\nu}{n+1}. \mid \int_{t_k}^{t_{k+1}} K(t_{k+1}, s)[f_y(s) - f_z(s)]ds| < \frac{\nu}{5(n+1)}.$$

Define

$$z(t) = K(t, t_0)y_0 + \int_{t_0}^{t} f_z(s)ds.$$

Consequently

$$|y(t) - z(t)| < |y(t_k) - z(t_k)| + |y(t) - y(t_k)| + |z(t) - z(t_k)|$$

$$\leq \frac{\nu}{5(n+1)}(n+1) + \frac{\nu}{5} + \frac{\nu}{5} \leq \nu,$$

i.e. $|y(t) - z(t)| \leq \nu$ and hence $y(\cdot)$ is $(\varepsilon + \nu)$-solution for which $|y(t) - z(t)| < \nu$. Taking $\nu < \delta$ we have that for any δ and any ν there exists $(\varepsilon + \delta)$-solution $z(\cdot)$ with $|y(t) - z(t)| < \nu$.

Consider the so called relaxed system

$$\begin{cases} \dot{y} \in A(t)y + \overline{co} \quad F(t, y(t)), \\ \qquad y(t_0) = y_0. \end{cases} \tag{2.10}$$

The ε-solutions and limit solutions are defined analogously.

It follows from Proposition 2.11 that under **A1, F1, F2** the set of limit solutions of (1.1) and (2.10) coincide.

3 One Sided Perron Evolution Inclusion

In this chapter we introduce OSP condition and prove a variant of the lemma of Filippov–Pliss and a variant of the relaxation theorem.

Definition 3.1 *A function $w: I \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be Perron function if it is Caratheodory (measurable on t and continuous on the state variable), integrally bounded on the bounded sets, $w(t,0) = 0$ and the unique solution of the problem $\dot{r}(t) = w(t, r(t))$, $r(t_0) = 0$ is $r(t) = 0$.*

The multifunction $F: I \times Y \rightrightarrows Y$ is called one-sided Perron (with respect to Perron function w) if for every $y_1, y_2 \in Y$, a.e. $t \in I$ and every $f \in F(t, y_1)$ there exist $g \in F(t, y_2)$ such that

$$[y_1 - y_2, f - g]_+ \leq w(t, \| y_1 - y_2 \|)$$

.

In this section we study the problem (1.1) when $F(t,\cdot)$ is one sided Perron (OSP). First we prove variant the well known Lemma of Fillipov-Pliss. Afterward we prove the existence of solutions for almost lower semicontinuous case and the relaxation theorem. Finally we prove the existence of limit solutions in case of OSP multifunction $F(t,\cdot)$ when $A(t)$ generates not necessarily compact evolution operator. We assume further in this section that $F(\cdot,\cdot)$ is almost continuous.

Theorem 3.2 *Let **A1, A2, F1, F2** holds. If $F(\cdot,\cdot)$ is almost LSC with nonempty closed valued, then the set of (mild) solutions of (1.1) is nonempty and $C(I,Y)$ precompact.*

Proof. Consider the following evolution inclusion

$$\begin{cases} \dot{z} \in A(t)z + N.\mathbb{B}, \\ \quad z(t_0) = y_0 \end{cases} \tag{3.1}$$

Using the same method as in the proof of Lemma 2.4 one can show that its solution set $Sol(N\mathbb{B})$ is $C(I,Y)$ precompact. Denote $C_k = \overline{co} \quad Sol(N\mathbb{B})$. Now we define the operator

$$\mathcal{L}(y(\cdot)) = \{f(\cdot): f(y) \in F(t, y(t)) \quad is \;\; measuarable\}.$$

Clearly $\mathcal{L}$ maps C_k into $L_1(I, Y)$. Since $F(\cdot,\cdot)$ is almost USC, one has that $\mathcal{L}(\cdot)$ is with nonempty closed values. Then it is easy to show , using Theorem 4.1 of [13] that $\mathcal{L}(\cdot)$ is LSC. Due to Bressan-Colombo's Theorem [3] there exists a selection $P(z) \in \mathcal{L}(z)$, which is a continuous map from C_k into $L_1(I, Y)$. Denote $P(w(\cdot))$ the solution of

$$\begin{cases} \dot{z} = A(t)z + P(w)(t), \\ \quad z(t_0) = y_0 \end{cases} \tag{3.2}$$

Clearly $P(p(\cdot))$ is continuous map from C_k into C_k. Due to Schauder's fixed point theorem there exists a fixed point $z(\cdot)$, i.e;

$$z(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)P(z)(s)ds.$$

Since $\| F(t, y) \| \leq N$, one has that $P(z)(s) \in F(s, z(s))$ and hence $z(\cdot)$ is a mild solution of (1.1).

Definition 3.3 *The continuous function $y(\cdot)$ is said to be outer ε-solution if*

$$\dot{y} \in A(t)y + F(t, y(t)) + g_\varepsilon(t)\mathbb{B},$$

where $g_\varepsilon(t) \geq 0$ with $\int_I g_\varepsilon(t)dt \leq \varepsilon$.

Due to growth condition and Lemma 2.3 we can assume that $|g_\varepsilon(t)| \leq N$. Let $F(\cdot,\cdot)$ be almost continuous. The following result then holds.

Theorem 3.4 *Under **F1, F2** and **A2**, for every $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that if $y(\cdot)$ is a δ-solution of (1.1) then it is an outer ε-solution of (1.1). Furthermore if $\delta(\varepsilon) \to 0 \Rightarrow \varepsilon \to 0$.*

Proof. The solution set of

$$\begin{cases} \dot{y} \in A(t)y + F(t, y + \mathbb{B}) + \mathbb{B}, \\ \qquad y(t_0) = y_0 \end{cases} \tag{3.3}$$

is $C(I, Y)$ precompact (cf. Remark 2.5). Thus the reachable set of (3.3) at the time t $Reach(t)$ is also precompact and hence $\Omega = \overline{\bigcup_{t \in [t_0, T]} Reach(t)}$ is compact. Since $F(\cdot,\cdot)$ is almost continuous $\Rightarrow \forall \upsilon > 0$ there exist compact $I_\upsilon \subset I$ with meas$(I_\upsilon) > (T - t_0) - \upsilon$ such that $F(\cdot,\cdot)$ is continuous and hence uniformly continuous on $I_\upsilon \times \Omega$. Fix $\varepsilon > 0$. Thus there exist $\delta > 0$ such that

$$D_H(F(t, x), F(t, (x + \delta\mathbb{B}) \cap \Omega) \leq \frac{\varepsilon}{(2(T - t_0) + 1)}.$$

Let $z(\cdot)$ be δ-solution

$$\dot{z} \in A(t)z + F(t, y(t) + \delta\mathbb{B}),$$

$$\Rightarrow \dot{z} = A(t)z + f_z(t), \quad f_z(t) \in F(t, z(t) + \delta\mathbb{B})$$

$$\Rightarrow f_z(t) \in F(t, z(t)) + \frac{\varepsilon}{(2(T - t_0) + 1)}$$

on $I_\upsilon \times Y$ and $\| f_z(t) \| \leq \mu$ on I_υ. Further

$$\int_I dist(f_z(t), F(t, y(t))dt \leq 2\mu\upsilon + \frac{\varepsilon}{2(T - t_0) + 1)}(T - t_0).$$

Hence $y(\cdot)$ is outer ε-solution when

$$2\mu\upsilon + \frac{\varepsilon}{2(T - t_0) + 1)}(T - t_0) \leq \varepsilon.$$

Lemma 3.5 *Assume **F1, F2, A2** holds. Moreover, suppose that $F(t,\cdot)$ is one-sided Perron with respect to Perron function $w(\cdot,\cdot)$. Let $h: I \to \mathbb{R}_+$ be a Lebesgue integrable function. If $y(\cdot)$ is a solution of*

$$\begin{cases} \dot{y} \in A(t)y + F(t, y(t)) + h(t)\mathbb{B}, \\ \qquad y(t_0) = y_0, \end{cases} \tag{3.4}$$

then, for every $\delta > 0$ there exist a solution $z(t)$ of (1.1) such that

$$|z(t) - y(t)| < r(t),$$

where $r(\cdot)$ is the maximal solution of

$$\begin{cases} \dot{r} = w(t, r(t)) + h(t) + \delta, \\ \quad r(t_0) = |z_0 - y_0|. \end{cases} \tag{3.5}$$

Proof. Clearly, $y(\cdot)$ is given by

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s)(f_y(s) + g_y(s))ds,$$

where $f_y(t) \in F(t, y(t))$ and $g_y(t) \in h(t)\mathbb{B}$ for a.a. $t \in [t_0, T]$. Fix $\delta > 0$ and define

$$F_\delta(t, u) = \overline{\{v \in F(t, u); \quad [y(t) - u, f_y(t) - v]_+ < w(t, |y(t) - u|) + \delta\}}.$$

The multifunction F_δ has nonempty values since $F(t,\cdot)$ is One-sided Perron. We shall prove that F_δ is almost lower semicontinuous. It is enough to show that

$$\tilde{F}_\delta(t, u) = \{v \in F(t, u); \quad [y(t) - u, f_y(t) - v]_+ < w(t, |y(t) - u|) + \delta\}$$

is almost lower semicontinuous. From the almost lower semicontinuity of F and Lusin's property of $f_y(\cdot)$, we have that for every $\varepsilon > 0$ there exist a compact set $I_\varepsilon \subset I$ with meas$(I \backslash I_\varepsilon) < \varepsilon$ such that $F(\cdot,\cdot)$ is lower semicontinuous on $I_\varepsilon \times Y$, $w(\cdot,\cdot)$ is continuous on $I_\varepsilon \times Y$ and $f_y(\cdot)$ is continuous on I_ε. Therefore it remains to show that $\tilde{F}_\delta$ is lower semicontinuous on $I_\varepsilon \times Y$. Let $(t, u) \in I_\varepsilon \times Y$, $l \in \tilde{F}_\delta(t, u)$ and let the sequence $(t_n, u_n)_n \subset I_\varepsilon \times Y$. There exist $\eta > 0$ such that

$$[y(t) - u, f_y(t) - l]_+ \leq w(t, |y(t) - u|) + \delta - \eta. \tag{3.6}$$

Since F is lower semicontinuous at (t, u), one has that there exist a sequence $(l_n)_n$ with $l_n \in F(t_n, u_n)$ such that $l_n \to l$. Since f_y is continuous and $[\cdot,\cdot]_+$ is upper semicontinuous, then

$$[y(t_n) - u_n, f - y(t_n) - l_n]_+ \leq [y(t) - u, f_y(t) - l]_+ + \eta/2 \tag{3.7}$$

for n large enough. Furthermore, $w(\cdot,\cdot)$ is continuous and hence

$$w(t, |y(t) - u|) \leq w(t_n, |y(t_n) - u_n|) + \eta/2 \tag{3.8}$$

for n large enough. It follows from (3.6), (3.7) and (3.8) that $l_n \in \tilde{F}_\delta(t_n, u_n)$ for n large enough. Thus $\tilde{F}_\delta$ is almost lower semicontinuous and hence $F_\delta(\cdot,\cdot)$ is also almost LSC. From Theorem 3.2 we know that the differential inclusion

$$\begin{cases} \dot{y} \in A(t)y + F_\delta(t, y(t)), \\ \qquad y(t_0) = y_0, \end{cases} \tag{3.9}$$

has a solution $y(\cdot)$ defined on $[t_0, T]$ and given by

$$y(t) = K(t, t_0)y_0 + \int_{t_0}^{t} K(t, s) f_y(s) ds, \tag{3.10}$$

where $f_y(t) \in F_\delta(t, y(t))$ for a.a. $t \in [t_0, T]$. Using the properties of $[\cdot,\cdot]_+$ it is easy to show that

$$[y(t) - z(t), f_y(t) + g_y(t) - f_z(t)]_+ \leq w(t, |y(t) - z(t)|) + h(t) + \delta,$$

for a.a. $t \in [t_0, T]$. Since

$$|y(t) - z(t)| \leq |y_0 - z_0| + \int_{t_0}^{t} [y(s) - z(s), f_y(s) + g_y(s) - f_z(s)]_+ ds$$

for every $t \in [t_0, T]$, one has that

$$|y(t) - z(t)| \leq |y_0 - z_0| + \int_{t_0}^{t} w(s, |y(s) - z(s)|) ds + \int_{t_0}^{t} h(s) ds + \delta t,$$

for every $t \in [t_0, T]$. Therefore we obtain that

$$|y(t) - z(t)| \leq r(t)$$

for any $t \in [t_0, T]$, where $r(\cdot)$ is the maximal solution of (3.5).

Theorem 3.6 *Assume **F1, F2** and **A2**. Moreover, suppose that $F(t,\cdot)$ is one sided Perron w.r.t the Perron function $w(\cdot,\cdot)$. Then the solution set of (1.1) is dense in the limit solution set of (2.10).*

Proof. Let $y(\cdot)$ be a limit solution of (2.10). Thus $\forall\delta > 0$ there exist $\frac{\delta}{2}$-solution $z_\delta(\cdot)$ such that

$$\| y(t) - z_\delta(t) \| \leq \frac{\delta}{2} \quad \forall t \in I.$$

Hence there exists δ-solution $z(\cdot)$ of (1.1) such that

$$\| z(t) - z_\delta(t) \| \leq \frac{\delta}{2}$$

and hence $\| y(t) - z(t) \| \leq \delta$. Let $\varepsilon > 0$. Then one can prove that $\delta = \delta(\varepsilon)$ is such that $z(\cdot)$ is outer ε-solution i.e. there exist $h(\cdot)$ with

$$\int_{t_0}^{T} h(t)dt \leq \varepsilon.$$

Due to the Lemma of Fillipovo-Pliss there exists a solution $\chi(\cdot)$ of such that

$$\| z(t) - \chi(t) \| \leq r(t),$$

where

$$\begin{cases} \dot{r} = w(t, r(t)) + h(t) + \frac{\delta}{2}, \\ \quad r(t_0) = |y_0 - \chi_0|. \end{cases} \tag{3.11}$$

Consequently

$$\| y(t) - \chi(t) \| \leq r(t) + \frac{3\delta}{2}.$$

The proof is complete, because $w(\cdot,\cdot)$ is a Perron function.

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